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Abstract

This paper presents a new, non-calculus approach to solving the utility maximization problem with CES utility function, as well as with Cobb-Douglas utility function in case of $n \geq 2$ commodities. Instead of using the Lagrange multiplier method or some other method based on differential calculus, these two maximization problems are solved by using Jensen's inequlity and weighted arithmetic-geometric mean (weighted AM-GM) inequality. In comparison with calculus methods, this approach does not require checking the first and the second order conditions.

Key words:
Utility maximization problem, CES and Cobb-Douglas utility function, mathematical inequalities, without calculus

JEL Classification
C69, D11

1. INTRODUCTION

In the last thirty years mathematical inequalities have been applied to various economic problems. In praticular, mathematical inequalities such as arithmetic-geometric mean inequality and Cauchy-Buniakowsky-Schwarz inequality, have benn used to solve many optimization problems in the field of inventory theory (see [2], [3], [8]). A good review of papers that deal with applications of mathematical inequalities to inventory theory models can be found in [3]. The most significant contribution of these papers is reflected in the fact that some important optimization problems in economics which cannot be trivially solved by using methods based on calculus are solved in a way much easier to understand, thus providing better insight into the nature of the problem. In this paper we consider two standard and very important microeconomic problems: the utility maximization problem with CES utility function and the utility maximization problem with Cobb-Douglas utility function. These two problems are usually solved by using differential calculus. However, standard microeconomic textbooks show solution only in the case of n=2 commodities (see for instance [5] and [7]). Therefore, the aim of this paper is to show how to solve these problems in an easier manner via mathematical inequalities for arbitrary $n\geq 2$. In this paper we use the following mathematical inequalities.

Theorem 1. (Jensen's inequality) Let n be a positive integer. If f is a convex function on $[a, b]$, then for any choice of $t_1,\ldots,t_n \in [0, 1]$ such that $\sum_{i=1}^{n} t_i = 1$, and for all $x_1,\ldots,x_n \in [a, b]$ the inequality

$$f\left(t_1x_1+\cdots+t_nx_n\right)\leq t_1 f\left(x_1\right)+\cdots t_n f\left(x_n\right) \tag{1}$$

holds. The equality in (1) holds if and only if $x_1=\ldots=x_n$. Note: if f is concave function, then in (1) reverse inequality holds.

Theorem 2. (Weighted AM-GM inequality) Let n be a positive integer. If $x_1,\ldots,x_n$ and $t_1,\ldots,t_n$ are positive numbers such that $\sum_{i=1}^{n} t_i = 1$, then

$$x_1^{t_1}\cdots x_n^{t_n}\leq t_1x_1+\cdots t_nx_n\,. \tag{2}$$

The equality in (2) holds if and only if $x_1=\ldots=x_n$.

Proofs of the previous theorems can be found, for instance, in [1] or [4].

2. CASE n=2

The consumer utility maximization problem can be stated as

$$\max_{x_1,\ldots,x_n\geq 0} u\left(x_1,\ldots,x_n\right) \tag{3}$$

$$\text{s. t.} \sum_{i=1}^{n} p_ix_i = y \tag{4}$$

where n is a positive integer and represents the number of commodities that consumer buys, $x_i\geq 0$ is the quantity of commodity i, $p_i>0$ is the price per unit of commodity i, $y>0$ is a consumer’s fixed money income and u is a strictly increasing and strictly quasiconcave utility function. If u is the CES utility function, then in case of n=2 commodities the problem (3)-(4) becomes the utility maximization problem with CES utility function (5)-(6):

$$\max_{x_1,x_2\geq 0} u^{CES}\left(x_1,x_2\right)=A\left(\alpha_1x_1^{\rho}+\alpha_2x_2^{\rho}\right)^{\frac{1}{\rho}} \tag{5}$$

$$\text{s. t. } p_1x_1+p_2x_2=y \tag{6}$$

where $\rho \in \langle -\infty, 1\rangle \setminus \{0\}$, coefficients $0<\alpha_1,\alpha_2<1$ describe consumer preferences and A>0 is the scale of production. Similarly, if u is the Cobb-Douglas utility function, then in case of n=2 problem (3)-(4) becomes the utility maximization problem with Cobb-Douglas utility function (7)-(8):

$$\max_{x_1,x_2\geq 0} u^{C-D}\left(x_1,x_2\right)=Ax_1^{\alpha_1}x_2^{\alpha_2} \tag{7}$$

$$\text{s. t. } p_1x_1+p_2x_2=y. \tag{8}$$

Theorem 3. The maximum utility in problem (5)-(6) is equal to

$$u_{\max}^{CES}=Ay\left(\alpha_1^{\frac{1}{1-\rho}}p_1^{\frac{1}{1-\rho}}+\alpha_2^{\frac{1}{1-\rho}}p_2^{\frac{1}{1-\rho}}\right)^{\frac{1-\rho}{\rho}} \tag{9}$$

and it is achieved for the unique global maximizer $\left(x_1^{*,CES},x_2^{*,CES}\right)$ where

$$x_k^{*,CES}=\frac{y}{\sum_{i=1}^{2}p_i\left(\frac{p_i}{p_k}\cdot\frac{\alpha_k}{\alpha_i}\right)^{\frac{1}{\rho-1}}},\ k=1,2. \tag{10}$$

Proof. We proceed as in [6]. From (6) we get

$$x_2=\frac{y}{p_2}-\frac{p_1}{p_2}x_1. \tag{11}$$

Substituting (11) in (5), we transform the problem (5)-(6) into an unconstrained maximization problem

$$\max_{x_1\geq 0}\overline{u}^{CES}\left(x_1\right)=A\left(\alpha_1x_1^{\rho}+\alpha_2\left(\frac{y}{p_2}-\frac{p_1x_1}{p_2}\right)^{\rho}\right)^{\frac{1}{\rho}}, \tag{12}$$

where $\overline{u}^{CES}$ is a function of one variable. Let $\varepsilon>0,\ t_1=\frac{1}{1+\varepsilon},\ t_2=\frac{\varepsilon}{1+\varepsilon}$. Note that 0<$t_1$, t_2<1, t_1+t_2=1. Note that function $\overline{u}^{CES}$ from (12) can equivalently be written as

$$\begin{aligned}\overline{u}^{CES}\left(x_1\right)&=A\left(\left(1+\varepsilon\right)t_1\alpha_1x_1^{\rho}+\left(1+\varepsilon\right)t_2\frac{\alpha_2}{\varepsilon}\left(\frac{y}{p_2}-\frac{p_1}{p_2}x_1\right)^{\rho}\right)^{\frac{1}{\rho}}\\&=\left(1+\varepsilon\right)^{\frac{1}{\rho}}A\left(t_1\left(\alpha_1^{\frac{1}{\rho}}x_1\right)^{\rho}+\left(\left(\frac{\alpha_2}{\varepsilon}\right)^{\frac{1}{\rho}}\left(\frac{y}{p_2}-\frac{p_1}{p_2}x_1\right)\right)^{\rho}\right)^{\frac{1}{\rho}}.\end{aligned} \tag{13}$$

Let us consider functions $f,g:\langle 0,+\infty\rangle\to \mathrm{IR},\ f\left(x\right)=x^{\rho},\ g\left(x\right)=x^{\frac{1}{\rho}}$. It is trivial to show that if $\rho\in\langle -\infty,0\rangle$, then f is convex and g is decreasing. If $\rho\in\langle 0,1\rangle$, then f is concave and g is increasing. If we combine these facts with Theorem 1, then from (13) it follows

$$\overline{u}^{CES}\left(x_1\right)\leq\left(1+\varepsilon\right)^{\frac{1}{\rho}}A\left(t_1\alpha_1^{\frac{1}{\rho}}x_1+t_2\left(\frac{\alpha_2}{\varepsilon}\right)^{\frac{1}{\rho}}\left(\frac{y}{p_2}-\frac{p_1}{p_2}x_1\right)\right). \tag{14}$$

The equality in (14) holds if and only if $t_1\alpha_1^{\frac{1}{\rho}}x_1-t_2\left(\frac{\alpha_2}{\varepsilon}\right)^{\frac{1}{\rho}}\frac{p_1}{p_2}x_1=0$ for all x_1. This is possible if and only if

$$t_1\alpha_1^{\frac{1}{\rho}}=t_2\left(\frac{\alpha_2}{\varepsilon}\right)^{\frac{1}{\rho}}\frac{p_1}{p_2}\Leftrightarrow\frac{t_1}{t_2}=\left(\frac{\alpha_2}{\varepsilon\alpha_1}\right)^{\frac{1}{\rho}}\frac{p_1}{p_2}. \tag{15}$$

Since $t_1+t_2=1$ and $\frac{t_1}{t_2}=\frac{1}{\varepsilon}$, from (15) we obtain

$$\varepsilon=\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{\rho}}\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}, \tag{16}$$

$$t_1=\frac{1}{1+\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{1-\rho}}\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}},\ t_2=\frac{\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{1-\rho}}\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}}{1+\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{1-\rho}}\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}}. \tag{17}$$

If we substitute (16) and (17) into (14), we obtain the maximum level of utility in case of CES utility function $\overline{u}_{\max}^{CES}$, that is

$$u_{\max}^{CES}=Ay\left(\alpha_1^{\frac{1}{1-\rho}}p_1^{\frac{\rho}{\rho-1}}+\alpha_2^{\frac{1}{1-\rho}}p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}}. \tag{18}$$

We still need to obtain the unique level of commodity quantities x_1 and x_2 for which the global maximum is obtained. By applying Theorem 1 to (11), we obtain that the maximum utility level (18) is achieved if and only if

$$\alpha_1^{\frac{1}{\rho}}x_1=\left(\frac{\alpha_2}{\varepsilon}\right)^{\frac{1}{\rho}}\left(\frac{y}{p_2}-\frac{p_1}{p_2}x_1\right). \tag{19}$$

Now, from (11), (16), (17) and (19), the unique global maximizer $\left(x_1^{*,CES},x_2^{*,CES}\right)$ for the problem (5)-(6) can be easily obtained as

$$x_k^{*,CES}=\frac{y}{\sum_{i=1}^{2}p_i\left(\frac{p_i}{p_k}\cdot\frac{\alpha_k}{\alpha_i}\right)^{\frac{1}{\rho-1}}},\ k=1,2. \tag{20}$$

The results (18) and (20) have the same form as in [5]. **Q.E.D.**

Theorem 4. The maximum utility in problem (7)-(8) is equal to

$$u_{\max}^{C-D}=\frac{A\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}y^{\alpha_1+\alpha_2}}{\left(\alpha_1+\alpha_2\right)^{\alpha_1+\alpha_2}p_1^{\alpha_1}p_2^{\alpha_2}}, \tag{21}$$

and it is achieved for the unique global maximizer $\left(x_1^{*,C-D},x_2^{*,C-D}\right)$ where

$$x_k^{*,C-D}=\frac{\alpha_k y}{p_k\left(\alpha_1+\alpha_2\right)},\ k=1,2. \tag{22}$$

Proof. By substituting (11) in (7), we obtain the unconstrained maximization problem

$$\max_{x_1\ge 0}\overline{u}^{C-D}\left(x_1\right)=Ax_1^{\alpha_1}\left(\frac{y}{p_2}-\frac{p_1}{p_2}x_1\right)^{\alpha_2}. \tag{23}$$

Let us transform (23) in the following way:

$$\begin{aligned}\overline{u}^{C-D}(x_1) &= \frac{A}{p_2^{\alpha_2}}\left(x_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\right)^{\alpha_1+\alpha_2}\left(\left(y-p_1x_1\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}\right)^{\alpha_1+\alpha_2} \\ &= \frac{A}{p_1^{\alpha_1}p_2^{\alpha_2}}\left(\alpha_2^{-\frac{\alpha_1}{\alpha_1+\alpha_2}}\left(\alpha_2p_1x_1\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\alpha_1^{-\frac{\alpha_2}{\alpha_1+\alpha_2}}\left(\alpha_1y-\alpha_1p_1x_1\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}\right)^{\alpha_1+\alpha_2} \\ &= A\frac{\alpha_1^{-\alpha_2}\alpha_2^{-\alpha_1}}{p_1^{\alpha_1}p_2^{\alpha_2}}\left(\left(\alpha_2p_1x_1\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\left(\alpha_1y-\alpha_1p_1x_1\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}\right)^{\alpha_1+\alpha_2}.\end{aligned} \tag{24}$$

Since $\frac{\alpha_1}{\alpha_1+\alpha_2}+\frac{\alpha_2}{\alpha_1+\alpha_2}=1$, by applying Theorem 2 to (24), we obtain

$$\begin{aligned}\overline{u}^{C-D}(x_1) &\le A\frac{\alpha_1^{-\alpha_2}\alpha_2^{-\alpha_1}}{p_1^{\alpha_1}p_2^{\alpha_2}}\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\alpha_2p_1x_1+\frac{\alpha_2}{\alpha_1+\alpha_2}\alpha_1y-\frac{\alpha_2}{\alpha_1+\alpha_2}\alpha_1p_1x_1\right) \\ &= A\frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}{\left(\alpha_1+\alpha_2\right)^{\alpha_1+\alpha_2}}\cdot\frac{y^{\alpha_1+\alpha_2}}{p_1^{\alpha_1}p_2^{\alpha_2}}.\end{aligned} \tag{25}$$

Thus, by Theorem 2, the maximum utility level is equal to

$$u_{\max}^{C-D}=\frac{A\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}y^{\alpha_1+\alpha_2}}{\left(\alpha_1+\alpha_2\right)^{\alpha_1+\alpha_2}p_1^{\alpha_1}p_2^{\alpha_2}} \tag{26}$$

and it is achieved if and only if

$$\alpha_2p_1x_1=\alpha_1y-\alpha_1p_1x_1. \tag{27}$$

From (27) we get the unique optimal quantities of commodities 1 and 2, i.e. the unique global maximizer $\left(x_1^{*,C-D},x_2^{*,C-D}\right)$, where

$$x_1^{*,C-D}=\frac{\alpha_1y}{p_1\left(\alpha_1+\alpha_2\right)},\ x_2^{*,C-D}=\frac{\alpha_2y}{p_2\left(\alpha_1+\alpha_2\right)}. \tag{28}$$

Q.E.D.

One of the common assumption in economics is that utility function needs to be quasiconcave. In order for the Cobb-Douglas function to meet this condition, coefficients $\alpha_1,\alpha_2\ge 0$ have to satisfy the condition $\alpha_1+\alpha_2\le 1$. If $\alpha_1+\alpha_2=1$, that is if the Cobb-Douglas function exhibit constant returns to scale, then (26) and (28) can be simplified as

$$u_{\max}^{C-D}=Ay\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1}\left(\frac{\alpha_2}{p_2}\right)^{\alpha_2},\ x_1^{*,C-D}=\frac{\alpha_1y}{p_1},\ x_2^{*,C-D}=\frac{\alpha_2y}{p_2}. \tag{29}$$

Results (9), (10) and (29) are already known in contemporary microeconomic literature. However, to the best of my knowledge, this is the first time that these results are derived by using mathematical inequalities.

In the next section, the results are generalized for the case of n>2 commodities. To the best of my knowledge, the results obtained for the general case are unknown in contemporary economic and mathematical literature.

3. GENERAL CASE n>2

We consider the following utility maximization problem in case of CES utility function for n>2:

$$\max_{x_1,\dots,x_n\ge 0} u^{CES}\left(x_1,\dots,x_n\right)=A\left(\alpha_1x_1^{\rho}+\cdots+\alpha_nx_n^{\rho}\right)^{\frac{1}{\rho}} \tag{30}$$

$$\text{s. t. } p_1x_1+\cdots+p_nx_n=y. \tag{31}$$

Theorem 5. The maximum level of utility in problem (30)-(31) is equal to

$$u_{\max}^{CES} = Ay\left(\sum_{i=1}^{n}\alpha_i^{\frac{1}{1-\rho}} p_i^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} \tag{32}$$

and it is achieved for the unique global maximizer $\left(x_1^{*,CES},\dots,x_n^{*,CES}\right)$, where

$$x_k^{*,CES} = \frac{y}{\sum_{i=1}^{n} p_i\left(\frac{p_i}{p_k}\cdot\frac{\alpha_k}{\alpha_i}\right)^{\frac{1}{\rho-1}}},\ k=1,2,...,n. \tag{33}$$

Proof. We prove the theorem by using mathematical induction over the number of commodities $n\geq 2$.

(i) The claim of Theorem 5 holds for $n=2$, as shown in Theorem 3.

(ii) Assume that the claim of Theorem 5 holds for $k=2, 3, \dots, n$.

(iii) Let us prove that the claim of Theorem 5 holds for $k=n+1$.

Consider the following utility maximization problem

$$\max_{x_1,\dots,x_{n+1}\geq 0} u^{CES}\left(x_1,...,x_{n+1}\right) = A\left(\alpha_1 x_1^{\rho} + \cdots \alpha_{n+1}x_{n+1}^{\rho}\right)^{\frac{1}{\rho}} \tag{34}$$

$$\text{s. t. } p_1 x_1 + \cdots + p_{n+1}x_{n+1} = y\ . \tag{35}$$

From (35) we obtain

$$x_{n+1} = \frac{y}{p_{n+1}} - \frac{p_1}{p_{n+1}}x_1 - \cdots - \frac{p_n}{p_{n+1}}x_n\ . \tag{36}$$

Substituting (36) into (35) we obtain

$$u^{CES}\left(x_1,\dots x_{n+1}\right) = A\left(\alpha_1 x_1^{\rho} + \cdots \alpha_n x_n^{\rho} + \alpha_{n+1}\left(\frac{\overline{x}_n}{p_{n+1}} - \frac{p_n}{p_{n+1}}x_n\right)^{\rho}\right)^{\frac{1}{\rho}} \tag{37}$$

where $\overline{x}_n = y - p_1 x_1 - \cdots - p_{n-1}x_{n-1}$. By the same reasoning as when obtaining (12) and (18), we apply the claim (i) to terms $\alpha_n x_n^{\rho}$ and $\alpha_{n+1}\left(\frac{\overline{x}_n}{p_{n+1}} - \frac{p_n}{p_{n+1}}x_n\right)^{\rho}$ from (37) and thus obtain

$$\begin{aligned} u^{CES}\left(x_1,\dots,x_{n+1}\right) &\leq A\left(\alpha_1 x_1^{\rho} + \cdots + \alpha_{n-1}x_{n-1}^{\rho} + \overline{x}_n^{\rho}\left(\alpha_n^{\frac{1}{1-\rho}} p_n^{\frac{\rho}{\rho-1}} + \alpha_{n+1}^{\frac{1}{1-\rho}} p_{n+1}^{\frac{\rho}{\rho-1}}\right)^{1-\rho}\right)^{\frac{1}{\rho}} \\ &= A\left(\alpha_1 x_1^{\rho} + \cdots + \alpha_{n-1}x_{n-1}^{\rho} + \overline{\alpha}_n \overline{x}_n^{\rho}\right)^{\frac{1}{\rho}}, \end{aligned} \tag{38}$$

where $\overline{\alpha}_n = \left(\alpha_n^{\frac{1}{1-\rho}} p_n^{\frac{\rho}{\rho-1}} + \alpha_{n+1}^{\frac{1}{1-\rho}} p_{n+1}^{\frac{\rho}{\rho-1}}\right)^{1-\rho}$. Note that maximizing the expression on the left hand side of (38) is equivalent to the following problem:

$$\max_{x_1,\dots,x_{n-1},\overline{x}_n\geq 0} u^{CES}\left(x_1,...,x_{n-1},\overline{x}_n\right) = A\left(\alpha_1 x_1^{\rho} + \cdots + \alpha_{n-1}x_{n-1}^{\rho} + \overline{\alpha}_n \overline{x}_n^{\rho}\right)^{\frac{1}{\rho}} \tag{39}$$

$$s.t.\ p_1 x_1 + \dots + p_{n-1}x_{n-1} + 1\cdot\overline{x}_n = y\,. \tag{40}$$

By applying claim (ii) to (39)-(40), from (38) we get

$$\begin{aligned} u^{CES}\left(x_1,...,x_n,x_{n+1}\right) &\leq Ay\left(\sum_{i=1}^{n-1}\alpha_i^{\frac{1}{1-\rho}} p_i^{\frac{\rho}{\rho-1}} + \overline{\alpha}_n^{\frac{1}{1-\rho}}\cdot 1^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} \\ &= Ay\left(\sum_{i=1}^{n+1}\alpha_i^{\frac{1}{1-\rho}} p_i^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}}. \end{aligned} \tag{41}$$

Now, the equality in (38) is achieved if and only if

$$x_n = \frac{\bar{x}_n}{p_n + p_{n+1}\left(\frac{p_{n+1}}{p_n}\cdot\frac{\alpha_n}{\alpha_{n+1}}\right)^{\frac{1}{\rho-1}}},\ x_{n+1} = \frac{\bar{x}_n}{p_n\left(\frac{p_n}{p_{n+1}}\cdot\frac{\alpha_{n+1}}{\alpha_n}\right)^{\frac{1}{\rho-1}} + p_{n+1}}\ . \tag{42}$$

Furthermore, equality in (41) is achieved if and only if

$$x_l = \frac{y}{\sum_{i=1}^{n-1} p_l\left(\frac{p_i\alpha_l}{p_l\alpha_i}\right)^{\frac{1}{\rho-1}} + 1\left(\frac{1\cdot\alpha_l}{p_l\bar{\alpha}_n}\right)^{\frac{1}{\rho-1}}} = \frac{y}{\sum_{i=1}^{n+1} p_l\left(\frac{p_i\alpha_l}{p_l\alpha_i}\right)^{\frac{1}{\rho-1}}}, \tag{43}$$

where l=1,2,…,n-1. For l=n we have

$$\bar{x}_n = \frac{y}{\sum_{i=1}^{n-1} p_i\left(\frac{p_i\bar{\alpha}_n}{1\cdot\alpha_i}\right)^{\frac{1}{\rho-1}} + 1} = y\cdot\frac{\alpha_n^{\frac{1}{1-\rho}} p_n^{\frac{\rho}{\rho-1}} + \alpha_{n+1}^{\frac{1}{1-\rho}} p_{n+1}^{\frac{\rho}{\rho-1}}}{\sum_{i=1}^{n-1} p_i\left(\frac{p_i}{\alpha_i}\right)^{\frac{1}{\rho-1}} + \alpha_n^{\frac{1}{1-\rho}} p_n^{\frac{\rho}{\rho-1}} + \alpha_{n+1}^{\frac{1}{1-\rho}} p_{n+1}^{\frac{\rho}{\rho-1}}}. \tag{44}$$

Combining (42) and (44) we get

$$x_n = \frac{y}{\sum_{i=1}^{n+1} p_i\left(\frac{p_i\alpha_n}{p_n\alpha_i}\right)^{\frac{1}{\rho-1}}},\ x_{n+1} = \frac{y}{\sum_{i=1}^{n+1} p_i\left(\frac{p_i\alpha_{n+1}}{p_{n+1}\alpha_i}\right)^{\frac{1}{\rho-1}}}\ . \tag{45}$$

Since (41), (43) and (45) prove the claim (iii), Theorem 5 is completely proved. **Q.E.D.**

Let us now consider the utility maximization problem in case of Cobb-Douglas utility function for n>2:

$$\max_{x_1,\dots,x_n\ge 0} u^{C-D}\left(x_1,\dots,x_n\right) = Ax_1^{\alpha_1}\cdots x_n^{\alpha_n}, \tag{46}$$

$$\text{s. t. } p_1x_1 + \cdots + p_nx_n = y\ . \tag{47}$$

Theorem 6. The maximum utility in problem (46)-(47) is equal to

$$u_{\max}^{C-D} = A\left(\frac{y}{\sum_{i=1}^{n}\alpha_i}\right)^{\sum_{i=1}^{n}\alpha_i}\prod_{i=1}^{n}\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}, \tag{48}$$

and it is achieved for the unique global maximizer $\left(x_1^{*,C-D},\dots,x_n^{*,C-D}\right)$, where

$$x_k^{*,C-D} = \frac{\alpha_k y}{p_k\sum_{i=1}^{n}\alpha_i},\ k = 1,2,\dots,n. \tag{49}$$

Note: If $\sum_{i=1}^{n}\alpha_i = 1$, then (48) and (49) can be written in a simplified form as

$$u_{\max}^{C-D} = Ay\prod_{i=1}^{n}\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}, \tag{50}$$

$$x_k^{*,C-D} = \frac{\alpha_k y}{p_k},\ k = 1,2,\dots,n. \tag{51}$$

Proof. By mathematical induction over n≥2. The idea is the same as in the proof of Theorem 5, where the claim (i) holds as shown in Theorem 4. **Q.E.D.**

Remark 1. If $\sum_{i=1}^{n}\alpha_i = 1$, then Cobb-Douglas utility function is a limit of CES utility function as ρ approaches to zero, i.e.

$$\lim_{\rho\to 0} A\left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right)^{\frac{1}{\rho}} = A\prod_{i=1}^{n} x_i^{\alpha_i} \ . \tag{52}$$

Proof. Let

$$L = \lim_{\rho\to 0}\left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right)^{\frac{1}{\rho}} \ . \tag{53}$$

By taking logarithm of (53) and by using the L'Hospital rule, we get

$$\ln L = \lim_{\rho\to 0}\frac{\ln\left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right)}{\rho} = \left(\frac{0}{0}\right) \overset{L'H}{=} \lim_{\rho\to 0}\frac{\sum_{i=1}^{n}\alpha_i x_i^{\rho}\ln x_i}{\sum_{i=1}^{n}\alpha_i x_i^{\rho}} = \frac{\sum_{i=1}^{n}\alpha_i \ln x_i}{\sum_{i=1}^{n}\alpha_i} = \ln\prod_{i=1}^{n} x_i^{\alpha_i} \ . \tag{54}$$

Thus, $L = \prod_{i=1}^{n} x_i^{\alpha_i}$ which proves Remark 1. **Q.E.D.**

Remark 2. If $\sum_{i=1}^{n}\alpha_i = 1$, then (50) is a limit of (32) as ρ approaches to zero, i.e.

$$\lim_{\rho\to 0} Ay\left(\sum_{i=1}^{n}\alpha_i^{\frac{1}{1-\rho}} p_i^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} = Ay\prod_{i=1}^{n}\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i} \ . \tag{55}$$

Proof. It is sufficient to prove that

$$L = \lim_{\rho\to 0}\left(\sum_{i=1}^{n}\alpha_i^{\frac{1}{1-\rho}} p_i^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} = \prod_{i=1}^{n}\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i} \ . \tag{56}$$

By taking logarithm and by using L'Hospital rule, we get

$$\begin{aligned}\ln L &= \lim_{\rho\to 0}\frac{\ln\left(\sum_{i=1}^{n}\alpha_i^{\frac{1}{1-\rho}} p_i^{\frac{\rho}{\rho-1}}\right)}{\frac{\rho}{\rho-1}} = \left(\frac{0}{0}\right) \overset{L'H}{=} \lim_{\rho\to 0}\frac{\sum_{i=1}^{n}\alpha_i^{\frac{1}{1-\rho}} p_i^{\frac{\rho}{\rho-1}}\frac{1}{(1-\rho)^2}\ln\left(\frac{\alpha_i}{p_i}\right)}{\left(\sum_{i=1}^{n}\alpha_i^{\frac{1}{1-\rho}} p_i^{\frac{\rho}{\rho-1}}\right)\cdot\frac{1}{(1-\rho)^2}} \\ &= \frac{\sum_{i=1}^{n}\alpha_i\ln\left(\frac{\alpha_i}{p_i}\right)}{\sum_{i=1}^{n}\alpha_i} = \sum_{i=1}^{n}\ln\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i} = \ln\prod_{i=1}^{n}\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}.\end{aligned} \tag{57}$$

Thus, $L = \prod_{i=1}^{n}\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}$. This proves Remark 2. **Q.E.D.**

Remark 3. If $\sum_{i=1}^{n}\alpha_i = 1$, then (51) is a limit of (33) as ρ approaches to zero, i.e.

$$\lim_{\rho\to 0}\frac{y}{\sum_{i=1}^{n} p_i\left(\frac{p_i\alpha_k}{p_k\alpha_i}\right)^{\frac{1}{\rho-1}}} = \frac{\alpha_k y}{p_k}, \tag{58}$$

for all k=1,2,…,n.
Proof.

$$\lim_{\rho\to 0}\frac{y}{\sum_{i=1}^{n} p_i\left(\frac{p_i\alpha_k}{p_k\alpha_i}\right)^{\frac{1}{\rho-1}}}=\frac{y}{\sum_{i=1}^{n} p_i\cdot\frac{p_k}{p_i}\cdot\frac{\alpha_i}{\alpha_k}}=\frac{\alpha_k y}{p_k}, \tag{59}$$

for all k=1,2,…,n.

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