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Solving the production cost minimization problem with the Cobb – Douglas production function without the use of derivatives

Abstract

In this paper, we propose a new original method to solve the production cost minimization problem with Cobb-Douglas production function by using the weighted arithmetic-geometric-mean inequality (weighted AM-GM inequality). Instead of using derivatives or the Lagrange multiplier method, the minimum costs and global minimizers in the case of the Cobb-Douglas production function are derived in the direct way. The result is first derived for the case of two inputs and then generalized for the problem with *n* inputs.

Key words:

global optimization, Cobb-Douglas technology, without derivatives, arithmetic mean, geometric mean

JEL Classification

C61, C65, D24

Rješavanje problema minimizacije troškova u slučaju Cobb – Douglasove funkcije proizvodnje bez upotrebe derivacija

Sažetak

U ovom radu predložena je nova, originalna metoda za rješavanje problema minimizacije troškova u slučaju Cobb – Douglasove funkcije proizvodnje upotrebom težinske nejednakosti između aritmetičke i geometrijske sredine (težinska AG nejednakost). Umjesto upotrebe derivacija ili metode Lagrangeovog množitelja, vrijednost minimalnih troškova u slučaju Cobb – Douglasove funkcije proizvodnje, te vrijednosti u kojima se ti troškovi postižu, izvedeni su na izravan način. Rezultat je najprije pokazan u slučaju dva inputa, a potom je generaliziran na slučaj od *n* inputa.

Ključne riječi:

globalna optimizacija, Cobb-Douglas tehnologija, bez upotrebe derivacija, aritmetička sredina, geometreijska sredina

JEL klasifikacija

C61, C65, D24

1. Introduction

The interest for the study of optimization techniques without the use of derivatives has been present in the economic literature for several decades now. There are a number of papers dealing with optimization problems in inventory theory, such as economic order quantity (EOQ) and economic production quantity (EPQ) models, or those dealing with the methods such as algebraic optimization methods, inequality methods and others (see for example (Cárdenas-Barrón ,2010), (Chiu et al., 2011), (Chiu, 2012), (Hseih et al., 2008), (Leung, 2012), (Ouyang et al., 2012), (Teng, 2009)). A good historical review of methods for solving optimization problems in inventory theory without the use of derivatives can be found in (Cárdenas-Barrón, 2011). The importance of such alternative approaches is mainly reflected in the fact that problems may be solved in a different, easier manner. Also, solving optimization problems without the use of derivatives allows the presentation of given problems to a wider audience, not necessarily with a background in calculus. Therefore, in this article we propose an alternative approach to solving the production cost minimization problem in the case of the Cobb - Douglas production function. Although this problem appears in many textbooks (for example (Mas-Colell et al., 1995), (Jehle, A. G., Reny P. J., 2011)), the solution of this problem is usually obtained by using the Lagrange multiplier method and therefore requires the use of calculus. In this paper we propose a new method based on the weighted arithmetic-geometric-mean inequality (weighted AM-GM inequality) to solve the problem. To the best of our knowledge, no one has ever applied this approach to solving the production costs minimization problem in the case of the Cobb-Douglas production function, so this is the first such result.

The rest of the paper is organized as follows. The notation is introduced in the second section, while the formulation of the problem in case of the Cobb-Douglas production function with two inputs is given in the third section. The fourth section proposes the method based on weighted AM-GM inequality for solving the problem with two inputs. The generalization of the result for minimization of production costs in case of the Cobb-Douglas production function with *n* inputs is given in the fifth section. Finally, the concluding remarks are given in the sixth section.

2. Notation

In this paper we use the same notation as in (Mas-Colell et al., 1995), that is:

$z_1, z_2, \ldots, z_n$	nonnegative real numbers, inputs
$w_1, w_2, \ldots, w_n$	input prices
q	chosen output level (or the amount of output)
$\alpha, \beta, \alpha_1, \alpha_2, \ldots, \alpha_n$	nonnegative real numbers, output elasticities of inputs.

3. Problem formulation

The production cost minimization problem in the case of the Cobb-Douglas production function with **two inputs** can be stated as the following constrained optimization problem:

$$\min_{z_1, z_2 \geq 0} \quad w_1 z_1 + w_2 z_2 \tag{1}$$

$$\text{subject to } \ q = f\left(z_1, z_2\right) = z_1^{\alpha} z_2^{\beta}, \tag{2}$$

where (1) represents the objective function (cost of inputs), while (2) represents the given level of outputs.

4. The case of two inputs

There are two common methods for solving the problem (1) - (2) known in the literature (see for example (Mas-Colell, 1995), (Varian, 2010), etc.). Namely, they are the substitution method and the Lagrange multiplier method. Both of these methods use differential calculus. The method we propose uses substitution, however instead of using derivatives, it uses the weighted AM-GM inequality:

Theorem 1. (Weighted AM-GM inequality) *Let* $a_1, a_2, \ldots, a_n$ *be nonnegative real numbers and let* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *be nonnegative real numbers such that* $\sum_{i=1}^{n} \lambda_i = 1$. *Then*

$$\sum_{i=1}^{n} \lambda_i a_i \geq \prod_{i=1}^{n} a_i^{\lambda_i}. \tag{3}$$

Equality in (3) holds if and only if $a_i = a_j$ *for all i, j such that* $\lambda_i, \lambda_j \neq 0$.

The proof of Theorem 1 can be found in for example (Bulajich et al., 2010) or (Hung, 2007).

Let us now apply the weighted AM-GM inequality to the problem (1)-(2). By using (3) we obtain

$$\lambda_1 a_1 + \lambda_2 a_2 \geq a_1^{\lambda_1} a_2^{\lambda_2}. \tag{4}$$

Furthermore, from (2) we get

$$z_2 = q^{\frac{1}{\beta}} z_1^{\frac{-\alpha}{\beta}}. \tag{5}$$

By substituting (5) into (1), we obtain the following unconstrained optimization problem:

$$\min_{z_1 \geq 0} \; f(z_1) = w_1 z_1 + w_2 q^{\frac{1}{\beta}} z_1^{\frac{-\alpha}{\beta}}. \tag{6}$$

Instead of finding the stationary points of function $f(z_1)$ by using the first derivative, we use the inequality (4). Namely, the function $f(z_1)$ from (6) can be equivalently written as

$$f(z_1) = \left(\frac{\alpha}{\beta} + 1\right) \left[\frac{\frac{\alpha}{\beta}}{\frac{\alpha}{\beta} + 1} \cdot \frac{w_1 z_1}{\frac{\alpha}{\beta}} + \frac{1}{\frac{\alpha}{\beta} + 1} \cdot w_2 q^{\frac{1}{\beta}} z_1^{\frac{-\alpha}{\beta}} \right]. \tag{7}$$

Let $a_1 = \frac{w_1 z_1}{\frac{\alpha}{\beta}}$, $a_2 = w_2 q^{\frac{1}{\beta}} z_1^{\frac{-\alpha}{\beta}}$, $\lambda_1 = \frac{\frac{\alpha}{\beta}}{\frac{\alpha}{\beta} + 1}$ and $\lambda_2 = \frac{1}{\frac{\alpha}{\beta} + 1}$. Since $\lambda_1 + \lambda_2 = 1$, by applying inequality (4) to (7) we obtain

$$\begin{aligned} f(z_1) &\geq \left(\frac{\alpha}{\beta} + 1\right) \cdot \left(\frac{w_1 z_1}{\frac{\alpha}{\beta}} \right)^{\frac{\alpha}{\alpha+\beta}} \cdot \left(w_2 q^{\frac{1}{\beta}} z_1^{\frac{-\alpha}{\beta}} \right)^{\frac{\beta}{\alpha+\beta}} \\ &= q^{\frac{1}{\alpha+\beta}} w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}} \left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}} \right), \end{aligned} \tag{8}$$

or equivalently

$$f\left(z_1\right) \geq q^{\frac{1}{\alpha+\beta}} w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}} \left(\alpha+\beta\right)\left(\frac{1}{\alpha^{\alpha}\beta^{\beta}}\right)^{\frac{1}{\alpha+\beta}}. \tag{9}$$

The equality in (8) holds if and only if

$$\frac{w_1 z_1}{\frac{\alpha}{\beta}} = w_2 q^{\frac{1}{\beta}} z_1^{\frac{-\alpha}{\beta}}, \tag{10}$$

and therefore

$$z_1 = \left(\frac{\alpha}{\beta}\cdot\frac{w_2}{w_1}\right)^{\frac{\beta}{\alpha+\beta}} q^{\frac{1}{\alpha+\beta}}. \tag{11}$$

By substituting (11) into (5) we obtain

$$z_2 = \left(\frac{\beta}{\alpha}\cdot\frac{w_1}{w_2}\right)^{\frac{\alpha}{\alpha+\beta}} q^{\frac{1}{\alpha+\beta}}. \tag{12}$$

Thus, the minimum of production costs in the case of the Cobb-Douglas production function with two inputs is equal to

$$C\left(w_1, w_2, q\right) = q^{\frac{1}{\alpha+\beta}} w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}} \left(\alpha+\beta\right)\left(\frac{1}{\alpha^{\alpha}\beta^{\beta}}\right)^{\frac{1}{\alpha+\beta}}, \tag{13}$$

and it is achieved for the level of inputs $z_1\left(w_1, w_2, q\right)$ and $z_2\left(w_1, w_2, q\right)$ given by (11) and (12) respectively.

It is important to emphasize that weights λ_1 and λ_2 are chosen in a way that the right hand side of inequalities (8) and (9) does not depend on the level of input z_1. The choice of weights λ_1 and λ_2 is explained in the Appendix. Also, it is important to note that the weighted AM-GM inequality immediately gives us the minimum value of the production costs as well as the level of inputs for which the minimum costs are achieved in a single step, as opposed to use of derivatives.

Furthermore, the Cobb-Douglas production function often has a property of constant returns to scale, that is $\alpha+\beta=1$. In other words, if we double the level of inputs z_1 i z_2, the output level q will double as well. In case of constant returns to scale, relations (11), (12) and (13) are simplified, since by substituting $\beta=1-\alpha$ we obtain

$$z_1\left(w_1, w_2, q\right) = \left(\frac{\alpha}{1-\alpha}\cdot\frac{w_2}{w_1}\right)^{1-\alpha} q, \tag{14}$$

$$z_2\left(w_1, w_2, q\right) = \left(\frac{1-\alpha}{\alpha}\cdot\frac{w_1}{w_2}\right)^{\alpha} q, \tag{15}$$

$$C\left(w_1, w_2, q\right) = q\left(\frac{w_1}{\alpha}\right)^{\alpha}\left(\frac{w_2}{1-\alpha}\right)^{1-\alpha}. \tag{16}$$

5. General case

In this section we consider the generalization of the problem (1)-(2). Let us consider the Cobb-Douglas production function with n inputs z_1, z_2, ..., z_n having output elasticities α_1, α_2, ..., α_n respectively. We solve the following problem:

$$\min_{z_1,\dots,z_n\ge 0} f_n(z_1,z_2,\dots,z_n) = w_1 z_1 + w_2 z_2 + \dots + w_n z_n \tag{17}$$

$$\text{subject to } z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} = q\,. \tag{18}$$

Theorem 2. *The minimum of the problem (17) – (18) is equal to the*

$$C_n(w_1,w_2,\dots,w_n,q) = q^{\frac{1}{\sigma_n}} \prod_{i=1}^{n} \left(\frac{w_i}{\alpha_i}\right)^{\frac{\alpha_i}{\sigma_n}} \sum_{k=1}^{n} \alpha_k\,, \tag{19}$$

and it is achieved for the input levels

$$z_k^* = \left(q \prod_{\substack{i=1\\ i\neq k}}^{n} \left(\frac{\alpha_k}{w_k}\cdot\frac{w_i}{\alpha_i}\right)^{\alpha_i} \right)^{\frac{1}{\sigma_n}}, \quad k = 1,\dots,n \tag{20}$$

where $\sigma_n = \alpha_1 + \alpha_2 + \dots + \alpha_n$.

Proof. We prove the theorem by the use of mathematical induction over the number of inputs $n\ge 2$.

(i) The claim of Theorem 2 holds for n=2, as shown in section 4. From (13) it follows

$$C_2(w_1,w_2,q) = q^{\frac{1}{\alpha_1+\alpha_2}} \cdot w_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \cdot w_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}} \cdot (\alpha_1+\alpha_2) \left(\frac{1}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}} \right)^{\frac{1}{\alpha_1+\alpha_2}} \tag{21}$$

(ii) Assume that the claim of Theorem 2 holds for k=2,3,...,n.

(iii) Let us prove that the claim of Theorem 2 holds for k=n+1. From (18) we have

$$z_{n+1} = q^{\frac{1}{\alpha_{n+1}}} z_1^{\frac{-\alpha_1}{\alpha_{n+1}}} z_2^{\frac{-\alpha_2}{\alpha_{n+1}}} \cdots z_n^{\frac{-\alpha_n}{\alpha_{n+1}}}\,. \tag{22}$$

By applying (22) to objective function in (17) for n+1 inputs we obtain

$$f_{n+1}(z_1,z_2,\dots,z_n,z_{n+1}) = w_1 z_1 + w_2 z_2 + \dots + w_n z_n + w_{n+1} q^{\frac{1}{\alpha_{n+1}}} z_1^{\frac{-\alpha_1}{\alpha_{n+1}}} z_2^{\frac{-\alpha_2}{\alpha_{n+1}}} \cdots z_n^{\frac{-\alpha_n}{\alpha_{n+1}}}\,. \tag{23}$$

If we apply the weighted AM-GM inequality to terms $w_n z_n$ and $w_{n+1} q^{\frac{1}{\alpha_{n+1}}} z_1^{\frac{-\alpha_1}{\alpha_{n+1}}} z_2^{\frac{-\alpha_2}{\alpha_{n+1}}} \cdots z_n^{\frac{-\alpha_n}{\alpha_{n+1}}}$, from (23) it follows

$$f_{n+1}(z_1,z_2,...,z_n,z_{n+1}) \geq w_1 z_1 + w_2 z_2 + ... + w_{n-1} z_{n-1} +$$
$$+q^{\frac{1}{\alpha_n+\alpha_{n+1}}} w_n^{\frac{\alpha_n}{\alpha_n+\alpha_{n+1}}} \left(w_{n+1} z_1^{\frac{-\alpha_1}{\alpha_{n+1}}} z_2^{\frac{-\alpha_2}{\alpha_{n+1}}} \cdots z_{n-1}^{\frac{-\alpha_{n-1}}{\alpha_{n+1}}} \right)^{\frac{\alpha_{n+1}}{\alpha_n+\alpha_{n+1}}} \cdot (\alpha_n+\alpha_{n+1}) \left(\frac{1}{\alpha_n^{\alpha_n} \alpha_{n+1}^{\alpha_{n+1}}} \right)^{\frac{1}{\alpha_n+\alpha_{n+1}}} =$$
$$= w_1 z_1 + w_2 z_2 + ... + w_{n-1} z_{n-1} +$$
$$+q^{\frac{1}{\alpha_n+\alpha_{n+1}}} \left(w_n^{\frac{\alpha_n}{\alpha_n+\alpha_{n+1}}} w_{n+1}^{\frac{\alpha_{n+1}}{\alpha_n+\alpha_{n+1}}} \right) (\alpha_n+\alpha_{n+1}) \left(\frac{1}{\alpha_n^{\alpha_n} \alpha_{n+1}^{\alpha_{n+1}}} \right)^{\frac{1}{\alpha_n+\alpha_{n+1}}} z_1^{\frac{-\alpha_1}{\alpha_n+\alpha_{n+1}}} z_2^{\frac{-\alpha_2}{\alpha_n+\alpha_{n+1}}} \cdots z_{n-1}^{\frac{-\alpha_{n-1}}{\alpha_n+\alpha_{n+1}}}. \tag{24}$$

Note that the right hand side of (24) is equivalent to the problem (17) - (18), i.e. to the problem

$$\min_{z_1,...,z_n \geq 0} f_n(z_1,z_2,...,\bar{z}_n) = w_1 z_1 + w_2 z_2 + ... + \bar{w}_n \bar{z}_n \tag{25}$$

$$\text{subject to } z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_{n-1}^{\alpha_{n-1}} \bar{z}_n^{\bar{\alpha}_n} = q, \tag{26}$$

where $\bar{w}_n = w_n^{\frac{\alpha_n}{\alpha_n+\alpha_{n+1}}} w_{n+1}^{\frac{\alpha_{n+1}}{\alpha_n+\alpha_{n+1}}} (\alpha_n+\alpha_{n+1}) \left(\frac{1}{\alpha_n^{\alpha_n} \alpha_{n+1}^{\alpha_{n+1}}} \right)^{\frac{1}{\alpha_n+\alpha_{n+1}}}$, $\bar{\alpha}_n = \alpha_n + \alpha_{n+1}$ and

$\bar{z}_n = q^{\frac{1}{\bar{\alpha}_n}} z_1^{-\frac{\alpha_1}{\bar{\alpha}_n}} z_2^{-\frac{\alpha_2}{\bar{\alpha}_n}} \cdots z_{n-1}^{-\frac{\alpha_{n-1}}{\bar{\alpha}_n}}$.

By assumption (ii) of mathematical induction, the claim of Theorem 2 holds for $k=n$ and therefore we can apply it to the problem (25)-(26). It follows that

$$f_{n+1}(z_1,z_2,...,z_n,z_{n+1}) \geq q^{\frac{1}{\alpha_1+\alpha_2+\cdots+\alpha_{n+1}}} w_1^{\frac{\alpha_1}{\alpha_1+\alpha_2+\cdots+\alpha_{n+1}}} \cdots w_{n-1}^{\frac{\alpha_{n-1}}{\alpha_1+\alpha_2+\cdots+\alpha_{n+1}}} \cdot$$
$$\cdot \left(w_n^{\frac{\alpha_n}{\alpha_n+\alpha_{n+1}}} w_{n+1}^{\frac{\alpha_{n+1}}{\alpha_n+\alpha_{n+1}}} (\alpha_n+\alpha_{n+1}) \left(\frac{1}{\alpha_n^{\alpha_n} \alpha_{n+1}^{\alpha_{n+1}}} \right)^{\frac{1}{\alpha_n+\alpha_{n+1}}} \right)^{\frac{\alpha_n+\alpha_{n+1}}{\alpha_1+\alpha_2+\cdots+\alpha_{n+1}}} \cdot \tag{27}$$
$$\cdot (\alpha_1+\alpha_2+\cdots+\alpha_n+\alpha_{n+1}) \left(\frac{1}{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \cdots \alpha_{n-1}^{\alpha_{n-1}} (\alpha_n+\alpha_{n+1})^{\alpha_n+\alpha_{n+1}}} \right)^{\frac{1}{\alpha_1+\alpha_2+\cdots+\alpha_{n+1}}}.$$

Finally, by rearranging the right hand side of (27) we obtain

$$f_{n+1}(z_1,z_2,...,z_{n+1}) \geq q^{\frac{1}{\sigma_{n+1}}} \prod_{i=1}^{n+1} \left(\frac{w_i}{\alpha_i} \right)^{\frac{\alpha_i}{\sigma_{n+1}}} \cdot \sum_{k=1}^{n+1} \alpha_k = C_{n+1}(w_1,w_2,...,w_{n+1},q). \tag{28}$$

Note that the right hand side of (28) is achieved for level of inputs given by (20), since

$$f_{n+1}\left(z_1^*,z_2^*,\ldots,z_{n+1}^*\right)=\sum_{k=1}^{n+1} w_k z_k^* = \sum_{k=1}^{n+1} w_k \left(q^{\frac{1}{\sigma_{n+1}}} \prod_{\substack{i=1\\ i\neq k}}^{n+1} \alpha_k^{\frac{\alpha_i}{\sigma_{n+1}}} w_k^{\frac{-\alpha_i}{\sigma_{n+1}}} w_i^{\frac{\alpha_i}{\sigma_{n+1}}} \alpha_i^{\frac{-\alpha_i}{\sigma_{n+1}}} \right)=$$

$$= q^{\frac{1}{\sigma_{n+1}}} \sum_{k=1}^{n+1} \left[w_k^{1-\frac{\sigma_n-\alpha_k}{\sigma_{n+1}}} \left(\prod_{\substack{i=1\\ i\neq k}}^{n+1} w_i^{\frac{\alpha_i}{\sigma_{n+1}}} \right) \cdot \alpha_k^{\frac{\sigma_{n+1}-\alpha_k}{\sigma_{n+1}}} \cdot \left(\prod_{\substack{i=1\\ i\neq k}}^{n+1} \alpha_i^{\frac{-\alpha_i}{\sigma_{n+1}}} \right) \right] = \tag{29}$$

$$= q^{\frac{1}{\sigma_{n+1}}} \sum_{k=1}^{n+1} \left[\left(\prod_{i=1}^{n+1} w_i^{\frac{\alpha_i}{\sigma_{n+1}}} \right) \left(\prod_{i=1}^{n+1} \alpha_i^{\frac{-\alpha_i}{\sigma_{n+1}}} \right) \alpha_k \right] = q^{\frac{1}{\sigma_{n+1}}} \prod_{i=1}^{n+1} \left(\frac{w_i}{\alpha_i} \right)^{\frac{\alpha_i}{\sigma_{n+1}}} \cdot \sum_{k=1}^{n+1} \alpha_k =$$

$$= C_{n+1}\left(w_1,w_2,\ldots,w_{n+1},q\right).$$

It remains to show that the level of inputs given by (20) satisfy constraint (18):

$$\prod_{k=1}^{n+1}\left(z_k^*\right)^{\alpha_k} = \prod_{k=1}^{n+1} \left(\left[q \prod_{\substack{i=1\\ i\neq k}}^{n+1} \left(\frac{\alpha_k}{w_k} \cdot \frac{w_i}{\alpha_i} \right)^{\alpha_i} \right]^{\frac{\alpha_k}{\sigma_{n+1}}} \right) =$$

$$= \prod_{k=1}^{n+1} \prod_{\substack{i=1\\ i\neq k}}^{n+1} q^{\frac{\alpha_k}{\sigma_{n+1}}} \left(\frac{\alpha_k}{w_k} \cdot \frac{w_i}{\alpha_i} \right)^{\frac{\alpha_i \alpha_k}{\sigma_{n+1}}} = q^{\sum_{k=1}^{n+1} \frac{\alpha_k}{\sigma_{n+1}}} \prod_{k=1}^{n+1} \prod_{\substack{i=1\\ i\neq k}}^{n+1} \left(\frac{\alpha_k}{w_k} \cdot \frac{w_i}{\alpha_i} \right)^{\frac{\alpha_i \alpha_k}{\sigma_{n+1}}} = q. \tag{30}$$

This proves the Theorem. □

6. Conclusions

In this paper we presented a new original method for solving the production cost minimization problem in the case of the Cobb-Douglas production function. We solved the problem by using the weighted AM-GM inequality, which unlike the methods already known in the literature does not require the knowledge of calculus. Compared to methods that use derivatives, our method is simpler and directly computes the minimal production costs and the corresponding optimal level of inputs in one step only. Furthermore, this method of solving the constrained optimization problems does not require the knowledge of calculus and thus can provide an easier way of approaching important optimization problems in economics.

Appendix

Let us show that the function $f:\langle 0,+\infty\rangle \to R$, $f(x)=x+x^{-\alpha}$, where α>0 is a real number, achieves its minimum $(\alpha+1)\left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha+1}}$ at point $x_{\min}=\alpha^{\alpha+1}$.

This problem can easily be solved by the use of derivatives. However, let us derive this result by the use of AM-GM inequality.
Let

$$\lambda_1=\frac{\gamma_1}{\gamma_1+\gamma_2},\ \lambda_2=\frac{\gamma_2}{\gamma_1+\gamma_2} \tag{A1}$$

where γ_1, γ_2>0 are real numbers. Note that $\lambda_1+\lambda_2=1$. By directly applying the weighted AM-GM inequality we obtain

$$\begin{aligned} f(x)&=x+x^{-\alpha}=\frac{\gamma_1}{\gamma_1}x+\frac{\gamma_2}{\gamma_2}x^{-\alpha}= \\ &=(\gamma_1+\gamma_2)\left(\lambda_1\frac{x}{\gamma_1}+\lambda_2\frac{x^{-\alpha}}{\gamma_2}\right)\geq \\ &\geq(\gamma_1+\gamma_2)\left(\frac{x}{\gamma_1}\right)^{\lambda_1}\left(\frac{x^{-\alpha}}{\gamma_2}\right)^{\lambda_2}= \\ &=(\gamma_1+\gamma_2)\left(\frac{1}{\gamma_1}\right)^{\lambda_1}\left(\frac{1}{\gamma_2}\right)^{\lambda_2}\left(x^{\gamma_1-\alpha\gamma_2}\right)^{\frac{1}{\gamma_1+\gamma_2}}. \end{aligned} \tag{A2}$$

The goal is to make the term $x^{\gamma_1-\alpha\gamma_2}$ on the right hand side of (A2) constant, which is possible if and only if $\gamma_1-\alpha\gamma_2=0$. Therefore, we have

$$\gamma_1=\alpha\gamma_2, \tag{A3}$$

which together with (A1) gives

$$\lambda_1=\frac{\alpha}{\alpha+1},\ \lambda_2=\frac{1}{\alpha+1}. \tag{A4}$$

By substituting (A3) and (A4) into (A2), we obtain

$$f(x)=x+x^{-\alpha}\geq(\alpha+1)\left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha+1}}, \tag{A5}$$

which proves the claim. Finally, note that the equality in (A2) holds if and only if

$$\frac{x}{\gamma_1}=\frac{x^{-\alpha}}{\gamma_2}. \tag{A6}$$

From (A3) and (A6) it easily follows that $x_{\min}=\alpha^{\alpha+1}$.

References

Bulajich Manfrino, R., Gómez Ortega, J. A., Valdez Delgado, R. (2009), *Inequalities: A Mathematical Olympiad Approach*, Birkhäuser Verlag, Basel.

Cardenás-Barrón, L. E. (2010), "An easy method to derive EOQ and EPQ inventory models with backorders", *Computers and Mathematics with Applications*, 59, 948–952.

Cárdenas-Barrón, L. E. (2011), "The derivation of EOQ/EPQ inventory models with two backorders costs using analytic geometry and algebra", *Applied Mathematical Modelling*,35, 2394-2407.

Chiu, Y-S. P., Liu, S-C., Chiu, C-L., Chang, H-H. (2011), "Mathematical modeling for determining the replenishment policy for EMQ model with rework and multiple shipments", *Mathematical and Computer Modelling*, 54, 2165 – 2174.

Chiu, S-W., Chen, K-K., Chiu, Y-S. P., Ting, C-K. (2012), "Notes on mathematical modeling approach used to determine replenishment policy for the EMQ model with rework and multiple shipments", *Applied Mathematics Letters*, 25, 1964 – 1968.

Hseih, T.-P., Chang H.-J., Weng, M.-W., Dye, C.-Y. (2008), "A simple approach to an integrated single-vendor single-buyer inventory system with shortage", *Production Planning and Control: The Management of Operations*, 19(6): 601 – 604.

Hung, P. K. (2007), *Secrets in Inequalities*. GIL Publishing House.

Jehle, G. A., Reny, P. J. (2011), *Advanced Microeconomic Theory*, 3rd Edition, FT Prentice Hall.

Leung, K-N. F. (2012), "Using the complete squared method to solve and analyze "Optimal inventory policies when the quantity backorder is uncertain"", *International Journal of Combinatorial Optimization Problems and Informatics*, 3(3), 35 – 42.

Mas-Colell, A., Whinston, M. D., Green, J. R. (1995), *Microeconomic Theory*, Oxford University Press, New York.

Ouyang, L.-Y., Chang, C.-T., Shum, P. (2012), "The EOQ with defective items and partially permissible delay in payments linked to order quantity derived algebraically", *Central European Journal of Operations Research*, 20(1), 141 – 160.

Teng, J.-T. (2009), "A simple method to compute economic order quantities", *European Journal of Operational Research*, 198, 351 – 353.

Varian, H. L. (2010): *Intermediate Microeconomics: A Modern Approach*. W. W. Norton & Company, New York.